FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

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Stochastic Analysis (SS 2019)

Exercise sheet 1

Submission: 17.04.2019

Problem 1 (3 Points)

Let $(X_n)_{n\geq 0}$ be the simple random walk, i.e.

$$X_0 = 0, \qquad X_n = \sum_{i=1}^{n} Z_i, \qquad (Z_i)_{i\geq 1} \text{ i.i.d. with } \mathbb{P}[Z_i = 1] = \frac{1}{2} = \mathbb{P}[Z_i = -1].$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(Z_1, ..., Z_n)$, $n \geq 1$. Check (and prove) whether the following random times are stopping times.

(a) $\tau_1 = \max\{n \leq 6 \,|\, X_n = 3\}$

(b) $\tau_2 = \min\{n \geq 1 \,|\, X_n = 0\}$

(c) $\tau_3 = \min\{n \geq 3 \,|\, X_n = -2\} - 2$

Define $\min\{\emptyset\} = \infty$ and $\max\{\emptyset\} = 0$.

Problem 2 (2 Points)

Let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)$-martingale. Let $(H_n)_{n\geq 1}$ be $(\mathcal{F}_n)$-predictable. Show that if $X_n \in L^2$, $n \geq 0$, and $H_n \in L^2$, $n \geq 1$, then $\big(I_n^X(H)\big)_{n\geq 0}$ is an $(\mathcal{F}_n)$-martingale.

Problem 3 (4 Points)

Let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)$-martingale and τ an $(\mathcal{F}_n)$-stopping time. Show that

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

if

(a) $(X_n)_{n\geq 0}$ is bounded and $\tau < \infty$, a.s.

(b) τ is integrable and $|X_n - X_{n-1}| \leq K < \infty$, a.s., for all $n \geq 1$.

Problem 4 (2 Points)

Show that the random times $\tau_1, \tau_2, ...$ defined after Corollary 1.8 are stopping times.

Problem 5 **(4 Points)**

Let τ and ϑ be $(\mathcal{F}_n)$-stopping times. Show that

(a) $\mathcal{F}_\tau$ is a σ-algebra,

(b) if $\vartheta \leq \tau$, then $\mathcal{F}_\vartheta \subseteq \mathcal{F}_\tau$,

(c) if $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)$ and $\tau < \infty$, a.s., then X_τ is $\mathcal{F}_\tau$ - measurable.

Total: 15 Points

Terms of submission:

- Solutions can be submitted in groups of at most 2 students.
- Please submit at the beginning of the lecture or until 08:10 a.m. in room 3523, Ernst-Abbe-Platz 2.